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John Lee, Esq. LL.D. in the Chair.

Thomas William Burr, Esq., of Highbury,
was balloted for and duly elected a Fellow of the Society.

It may be as well to remind Fellows of the Society, that the pages of the *Monthly Notices* are open for the publication of all brief communications relating to astronomical science, except so far as regards planetary observations and ephemerides, the necessity for inserting which in this periodical is generally superseded by the earlier and more complete publication of intelligence of a similar kind in the *Astronomische Nachrichten.* It is manifest, however, that a few observed positions of any newly-discovered body, whether planet or comet, as well as a determination of the elements of its orbit, will serve to enhance the usefulness of the *Monthly Notices;* and it is scarcely necessary to add, that any contributions of this nature will be inserted with pleasure.

On a new Solution of the Problem of Planetary Perturbation.
By Professor Encke.

(*Extract of a Letter to the Astronomer Royal.*)

"The subject of the small planets has continued to occupy my attention so much, that I could not communicate with you until I had relieved my mind of this load of anxiety. I consider myself very fortunate on this account in being enabled, through the meritorious services of my excellent Dr. Brünnow, to transmit to you in the annexed paper the mean elements of *Flora*, with the perturbations of the same planet as produced by *Jupiter* and *Saturn.*

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From the mode in which the calculation of these elements has been worked out, I am induced to entertain a confident hope that we have it now in our power to compute the perturbations of the first power of the masses for all the small planets, not even excepting *Pallas*, and that the time bestowed on the calculations is not immoderately long.

" The perturbations produced by *Jupiter* upon *Flora*, which possesses the advantage of being pretty distant from the disturbing planet, but which, on the other hand, has an excentricity of 9° and an inclination of 6°, will admit of being computed in four weeks without any immoderate haste; and even with respect to those of *Pallas*, I would pledge myself, without hesitation, to compute them for *Jupiter* and *Saturn* in the course of a winter, if nothing unusual interfered with the discharge of my ordinary duties.

" On a former occasion I had the honour of transmitting to you a copy of a paper which I communicated to the Academy, and in which I proposed an indirect method. This was not suitable to the object in view, as any one may easily convince himself, since the approximations conducted to the required result only in a few rare cases. Still the proposed method has not been useless, since it has given me the form which is most convenient for calculation, of the integral of the assigned differential equation which always reappears in the problem. By a mere simple consideration of the combinations which present themselves in this inquiry, the following solution has been obtained.

" If the differential equation has the form

$$\frac{d\,d\xi}{d\,t^2} + \frac{k^2}{r^{03}}\,\xi = \Sigma\left(a_i^{i'}\ \cos\,(i\,\mathrm{M} - i'\,\mathrm{M}')\right),$$

in which the cosine may be exchanged for the sine, and M, M′, denote the mean anomalies of the disturbed and disturbing planets, the mean motions of which are represented by μ and μ'; then, in every case, will the co-efficient of cos $(i\,\mathrm{M} - i'\,\mathrm{M}')$ or of sin $(i\,\mathrm{M} - i'\,\mathrm{M}')$ be thus determined by integration :—

$$\text{Let}\qquad \begin{aligned} x^0 &= r^0 \cos v^0 = p_0 + \Sigma\,(p_m \cos m\,\mathrm{M}) \\ y^0 &= r^0 \sin v^0 = \qquad\ \ \Sigma\,(q_m \sin m\,\mathrm{M}). \end{aligned}$$

$$\text{Also let}\qquad \frac{1}{r\,\mu - i'\,\mu'} - \frac{1}{s\,\mu - i'\,\mu'} = [r, s]\,;$$

then the coefficient of cos $(i\,\mathrm{M} - i'\,\mathrm{M}')$ in the integral will have the following form :—

$$\begin{array}{llll} \vdots & \vdots & \vdots & \vdots \\ \dfrac{a_{i-3}^{i'}}{k\,\sqrt{p}} \Big\{ [i-1, i-2]\,\dfrac{p_1 q_2 - p_2 q_1}{4} & + [i, i-3]\,\dfrac{p_0 q_3}{2} & + [i+1, i-4]\,\dfrac{p_1 q_4 + p_4 q_1}{4} & \\ \qquad + [i+2, i-5]\,\dfrac{p_2 q_5 + p_5 q_2}{4} & + [i+3, i-6]\,\dfrac{p_3 q_6 + p_6 q_3}{4}\ \ldots\ldots \Big\} & & \end{array}$$

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$$+\frac{a_{i-2}^{i'}}{k\sqrt{p}}\left\{[i,i-2]\frac{p_0q_2}{2}+[i+1,i-3]\frac{p_1q_3+p_3q_1}{4}+[i+2,i-4]\frac{p_2q_4+p_4q_2}{4}\right.$$
$$\left.+[i+3,i-5]\frac{p_3q_5+p_5q_3}{4}\;\ldots\ldots\right\}$$

$$+\frac{a_{i-1}^{i'}}{k\sqrt{p}}\left\{[i,i-1]\frac{p_0q_1}{2}+[i+1,i-2]\frac{p_1q_2+p_2q_1}{4}+[i+2,i-3]\frac{p_2q_3+p_3q_2}{4}\right.$$
$$\left.+[i+3,i-4]\frac{p_3q_4+p_4q_3}{4}+[i+4,i-5]\frac{p_4q_5+p_5q_4}{4}\;\ldots\ldots\right\}$$

$$+\frac{a_{i}^{i'}}{k\sqrt{p}}\left\{[i+1,i-1]\frac{p_1q_1}{2}+[i+2,i-2]\frac{p_2q_2}{2}+[i+3,i-3]\frac{p_3q_3}{2}\right.$$
$$\left.+[i+4,i-4]\frac{p_4q_4}{4}\;\ldots\ldots\right\}$$

$$+\frac{a_{i+1}^{i'}}{k\sqrt{p}}\left\{[i+1,i]\frac{p_0q_1}{2}+[i+2,i-1]\frac{p_1q_2+p_2q_1}{4}+[i+3,i-2]\frac{p_2q_3+q_3p_2}{4}\right.$$
$$\left.+[i+4,i-3]\frac{p_3q_4+p_4q_3}{4}+[i+5,i-4]\frac{p_4q_5+p_5q_4}{4}\;\ldots\ldots\right\}$$

$$+\frac{a_{i+2}^{i'}}{k\sqrt{p}}\left\{[i+2,i]\frac{p_0q_2}{2}+[i+3,i-1]\frac{p_1q_3+p_3q_1}{4}+[i+4,i-2]\frac{p_2q_4+p_4q_2}{4}\right.$$
$$\left.+[i+5,i-3]\frac{p_3q_5+p_5q_3}{4}\;\ldots\ldots\right\}$$

$$+\frac{a_{i+3}^{i'}}{k\sqrt{p}}\left\{[i+2,i+1]\frac{p_1q_2-p_2q_1}{4}+[i+3,i]\frac{p_0q_3}{2}+[i+4,i-1]\frac{p_1q_4+p_4q_1}{4}\right.$$
$$\left.+[i+5,i-2]\frac{p_2q_5+p_5q_2}{4}+[i+6,i-3]\frac{p_3q_6+p_6q_3}{4}\;\ldots\ldots\right\}$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

" This expression proceeds in an infinite series of terms, both in the vertical and horizontal directions. It, however, converges very quickly for any given values of i and i', so that in general, only a very small number of coefficients require to be taken into consideration in order to determine the coefficient of the integral; and since, for the sine and cosine, the expression is merely a numerical quantity, when μ and μ' are given, so is the calculation for the very small number of values of i, when i' is constant, by no means difficult.

" Dr. Brünnow has succeeded in avoiding the indirect solution by adding to the equation relative to the undisturbed orbit, viz.:

$$\frac{1}{2}\,\frac{d^2(r_0^2)}{d\,t^2}=k^2\left(\frac{1}{r_0}-\frac{1}{a}\right);$$

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the equation for the disturbed orbit, viz.:

$$\tfrac{1}{2}\frac{d^2\,(r^2)}{dt^2} = k^2\left(\frac{1}{r}-\frac{1}{a}\right) + x\mathrm{X} + y\mathrm{Y} + z\mathrm{Z} + 2\int\left(\mathrm{X}\,\frac{dx}{dt} + \mathrm{Y}\,.\,\frac{dy}{dt} + \mathrm{Z}\,.\,\frac{dz}{dt}\right);$$

Whence, by comparison, there result the following rigorous equations:—

$$\tfrac{1}{2}\frac{d^2\,(r^2-r^{0\,2})}{dt^2} - \left(\frac{k^2}{r}-\frac{k^2}{r^0}\right) = x\mathrm{X} + y\mathrm{Y} + z\mathrm{Z} + 2\int(\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz),$$

$$\frac{d^2\,(x-x^0)}{dt^2} + \frac{k^2 x}{r^3} - \frac{k^2 x^0}{r^{0\,3}} = \mathrm{X}$$

$$\frac{d^2\,(y-y^0)}{dt^2} + \frac{k^2 y}{r^3} - \frac{k^2 y^0}{r^{0\,8}} = \mathrm{Y}$$

$$\frac{d^2\,(z-z^0)}{dt^2} + \frac{k^2 z}{r^3} - \frac{k^2 z^0}{r^{0\,3}} = \mathrm{Z}.$$

Let us now suppose

$$\begin{aligned} r &= r^0 + \delta r \\ x &= x^0 + \xi \\ y &= y^0 + \eta \\ z &= z^0 + \zeta, \end{aligned}$$

and let us collect together the squares and products of δr, ξ, η and ζ. We hence deduce these four equations:—

$$\frac{dd\,(r^0\,\delta r)}{dt^2} + \frac{k^2}{r^{0\,3}}(r^0\delta r) = x\mathrm{X} + y\mathrm{Y} + z\mathrm{Z} + 2\int\left(\mathrm{X}\,\frac{dx}{dt} + \mathrm{Y}\frac{dy}{dt} + \mathrm{Z}\frac{dz}{dt}\right)$$
$$- \tfrac{1}{2}\frac{dd\,(\delta r)^2}{dt^2} + \frac{k^2}{r^{0\,3}}(\delta r)^2 - \frac{k^2}{r^{0\,4}}(\delta r)^3$$

$$\frac{dd\xi}{dt^2} + \frac{k^2}{r^{0\,3}}\,\xi = \mathrm{X} + 3\,\frac{k^2 x^0}{r^{0\,5}}\,(r^0\,\delta r) - 6\,\frac{k^2 x^0}{r^{0\,7}}\,(r^0\,\delta r)^2$$
$$+ 3\,\frac{k^2\xi}{r^{0\,5}}\,(r^0\,\delta r)$$

$$\frac{dd\eta}{dt^2} + \frac{k^2}{r^{0\,3}}\,\eta = \mathrm{Y} + \frac{3k^2y^0}{r^{0\,5}}\,(r^0\,\delta r) - 6\,\frac{k^2 y^0}{r^{0\,7}}\,(r^0\,\delta r)^2$$
$$+ 3\,\frac{k^2\eta}{r^{0\,5}}\,(r^0\,\delta r)$$

$$\frac{dd\zeta}{dt^2} + \frac{k^2}{r^{0\,3}}\,\zeta = \mathrm{Z} + \frac{3\,k^2 z^0}{r^{0\,5}}\,(r^0\,\delta r) - 6\,\frac{k^2 z^0}{r^{0\,7}}\,(r^0\,\delta r)^2$$
$$+ 3\,\frac{k^2\zeta}{r^{0\,5}}\,(r^0\,\delta r)$$

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"The first of these equations gives $r^0 \delta r$, and this substituted in the others gives ξ, η, ζ.

"These are essentially the first equations of the *Mécanique Céleste.*"

"A second step due to Dr. Brünnow is that he has executed the numerical calculations of all the multiplications of the prescribed infinite series. This multiplication admits of being made with convenience and accuracy; and strictly affords the advantage of being far shorter than the substitutions would be if the numerical values for c were determined by multiplications executed analytically. M. Hansen had already recommended this mode of executing the numerical calculation without having further recourse to any analytical process; but it has not, so far as I know, been so generally applied in practice.

"These are the fundamental principles of the solution. Dr. Brünnow, however, has not executed the calculations absolutely in this form, but has introduced the polar co-ordinates r^0 and u^0, conformably to his method of treating the subject of special perturbations. It is quite possible, since the solution of the equations is thereby considerably simplified ($r^0 \delta r$ being already obtained by the first equation, and consequently there only remaining δu and z to be determined), that the abbreviation is so considerable as to outweigh the disadvantages attending the somewhat less elegant symmetry of the final equations. In the meanwhile, being curious to know to what extent the expressions for ξ, η, ζ, would be found to be less convergent than δu, I have also begun the calculation by the formula here given, and hope to be soon prepared with it; should the difference be not too great, the simple mode of deducing the formulæ for ξ, η, and ζ, will always captivate me. However, this is a mere matter of taste.

"We have both found the expression for the disturbing force by the actual developement of $\frac{1}{\varrho^3}$ (ϱ being the distance between the two planets), and then by multiplication have deduced the values of the resolved forces. We thereby derive the advantage of having fewer values of ϱ^3 to calculate throughout the entire extent of the periphery. Perhaps it would be better to develope $\frac{1}{\varrho}$ into a series according to powers of M and M′, and then by two multiplications to find $\frac{1}{\varrho^3}$.

"The terms of the second order admit of being easily obtained according to this method by means of Taylor's *Theorem*, since the quantities x, y, z, are only involved, the perturbations of which are already known. But this process of calculation, which in every case is very troublesome, would be useless when the question refers to the small planets, since the elements of these bodies are not known with sufficient accuracy. Dr. Brünnow, by means of his osculating elements, has obtained pretty accurate values of the disturbing forces; and by applying the resulting

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perturbations has deduced the mean elements of the planet. To me it is as good as certain that the perturbations thus calculated (the hundredths of seconds are omitted) are sufficiently accurate to be employed next year for a more exact determination of the mean elements. The terms of the first power of the masses which are here given will therefore subserve the object in view, and on a future occasion the question may be investigated, whether the terms of the second order exercise any sensible influence.

"The perturbational values are all given as they follow from the calculations. Moreover, the secular equations of the elements which might be deduced from the terms involving t (the unit of t is the Julian year) are not deduced, since for the next year the calculation in its present form is more convenient. By the way, I see that I have not made any mention of these terms in speaking of the integration of the differential equations. However, what I have said above will be quite sufficient to give you an idea of this method of computing the perturbations.

"*Berlin, April* 24, 1853."

The following are the results obtained by Dr. Brünnow relative to the planet *Flora*, to which Professor Encke alludes at the beginning of the foregoing extract:—

Mean Elements of Flora with the Perturbational Equations for Jupiter and Saturn. Calculated by Dr. Brünnow, Assistant at the Berlin Observatory, April 1853.

For the five normal places which were employed in the special perturbations, by means of the resulting perturbational equations for *Jupiter* and *Saturn*, purely elliptic mean elements were found. In this manner were obtained:—

Epoch 1848, Jan. 1·0, Mean Berlin Time.

	°	′	″	
Mean Anomaly	35	54	3·62	
Longitude of the Perihelion	32	54	28·34	Mean Equinox of the Epoch.
Long. Asc. Node	110	17	48·62	
Inclination	5	53	7·96	
Angle of Excentricity	9	0	56·29	
Mean, Dail. Sid. Mo.	1086″·330982			
Log. Sem. Maj.-axis	0·3426963			

These elements gave the mean place, and by applying the perturbational equations to each of the five normal places, the latter were thus represented:—